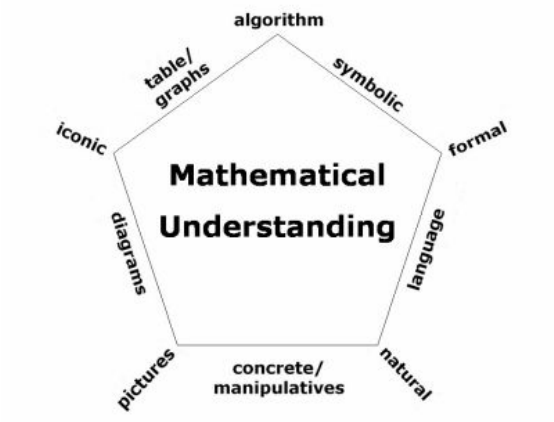
Arithmetic in Support of Algebra (and Geometry)

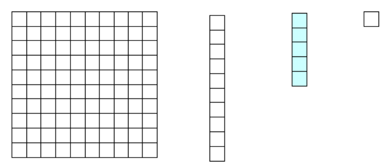
How do we *know* students know what we *think* we've taught 'em? Metaphors, iconic graphics, and multiple representations may be indicators. If students (or people!) know only one way of doing or thinking about something and/or can't really explain why they do it a certain way, we suspect that they don't really know what they're doing.



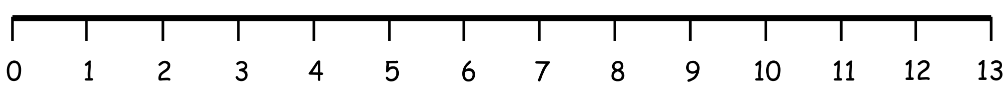
But when students can say something is *like* something else—say, that subtraction is like finding the missing addend you need to get a certain sum, or that multiplication and division relationships are like rectangles where the area is the product of the sides (factors). Or that counting forwards and backwards is like moving left or right on a number line. Or that a system of integers is just another number line going the other way from zero. Or that fractions are finer and finer partitions of the spaces between the units on a number line. Or that irrational numbers are numbers you can't ever quite pin down to a division, no matter how fine you get. Or that multiplication is like repeated addition is like hops on a number line is like patterns on a 10x10 grid is like stacks of one-by-whatever rows in a rectangular array. And so on.

Here's a loose collection of media and tools for multiple representations that may give you some ideas to explore or with plenty of graphics—if you grab this in .doc format—to snag and use.

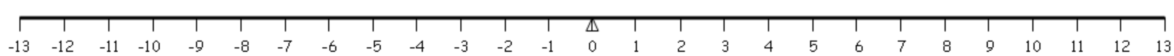
counting



- units (what are you counting)
- number names
- zero—what a concept!
- base ten system (composition and decomposition)
- stacking
  - dimension 0 = 1 cm x 1 cm x 1 cm unit cube
  - dimension 1 = 10 cm x 1 cm x 1 cm long .... length
  - dimension 2 = 10 cm x 10 cm x 1 cm flat .... area (length x width)
  - dimension 3 = 10 cm x 10 cm x 10 cm 1000-cube .... volume (length x width x height)
- number line: natural numbers (include zero)
  - a ray originating at zero



- number line: integers (left and right)
  - a line stretching infinitely in both + and − directions with zero at its midpoint

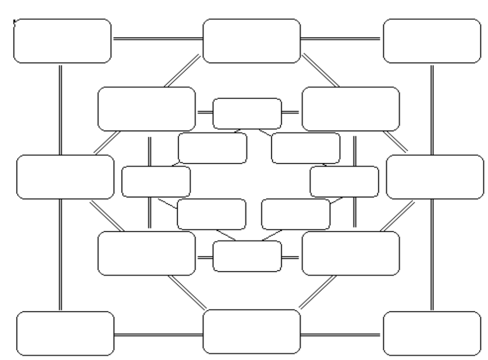


- construct a number line with compass and straightedge

adding and subtracting

Because a main goal of this course is teaching kids to compose, decompose, and recompose numbers and because all these kids have already been exposed to and learned something about basic operations in grades K-3 and because both are so closely related, addition and subtraction should be taught together.

- number line **combining**
  - 4 facts: 8+2 = 10, 2+8=10, 10-2=8, 10-8=2
- number line **comparing**
  - what's the difference between 10 and 2?
  - how much do you have to add to 2 to get 10? (missing addend)
  - and on the second pass: how much do you have to add to 8 to get 2?
- diffies: roll dice to build problems (see more on dice, below)
  - use 20-sided dice with first timers,
  - graduate to larger numbers: separate dice for ones, tens, even hundreds!
  - for online diffies, see the National Library of Virtual Manipulatives: http://nlvm.usu.edu/en/nav/frames_asid_326_g_2_t_1.html
- graph number combinations to 10: x + y = 10
- skip-counting (multiplication)
- race games: race up and back to 20, 100, 1000, 10,000
  - see an overview at http://www.soesd.k12.or.us/files/race_games.pdf
- fluency/automaticity: time tests with Holey Cards and used paper
- n − n = 0 additive inverse
- n + 0 = n identity element for addition



blank diffy

regrouping in base 10 (composition and decomposition)

trading in accumulations (stacks) of ten for the next higher unit
trading in a block for ten of the next lower type
counting strips
race games up and back to 20, to 100, to 1000, to 10,000
diffies

grids

addition facts, multiplication facts and skip counting
place value, numbers to 100 (then numbers to 120)

equal chunks of number lines can be stacked into grids—twos, fives, and tens are common and intuitive

1	**2**	3	**4**	5	**6**	7	**8**	9	**10**
11	**12**	13	**14**	15	**16**	17	**18**	19	**20**

1	2	**3**	4	5	**6**	7	8	**9**	10
11	**12**	13	14	**15**	16	17	**18**	19	20
21	22	23	**24**	25	26	**27**	28	29	**30**

1	2	3	**4**	5	6	7	**8**	9	10
11	**12**	13	14	15	**16**	17	18	19	**20**
21	22	23	**24**	25	26	27	**28**	29	30
31	**32**	33	34	35	**36**	37	38	39	**40**

1	2	3	4	**5**	6	7	8	9	**10**
11	12	13	14	**15**	16	17	18	19	**20**
21	22	23	24	**25**	26	27	28	29	**30**
31	32	33	34	**35**	36	37	38	39	**40**
41	42	43	44	**45**	46	47	48	49	**50**

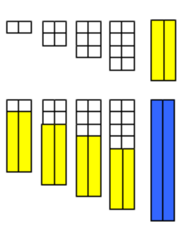
1	2	3	4	5	6	7	8	9	**10**
11	12	13	14	15	16	17	18	19	**20**
21	22	23	24	25	26	27	28	29	**30**
31	32	33	34	35	36	37	38	39	**40**
41	42	43	44	45	46	47	48	49	**50**
51	52	53	54	55	56	57	58	59	**60**
61	62	63	64	65	66	67	68	69	**70**
71	72	73	74	75	76	77	78	79	**80**
81	82	83	84	85	86	87	88	89	**90**
91	92	93	94	95	96	97	98	99	**100**

1	2	3	4	5	**6**	7	8	9	10
11	**12**	13	14	15	16	17	**18**	19	20
21	22	23	**24**	25	26	27	28	29	**30**
31	32	33	34	35	**36**	37	38	39	40
41	**42**	43	44	45	46	47	**48**	49	50
51	52	53	**54**	55	56	57	58	59	**60**

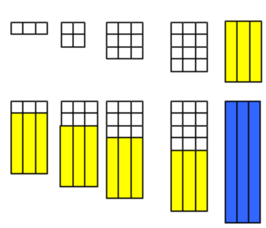
make your own grids in Excel or Word or download them from www.soesd.k12.or.us/math

arrays (The area model for multiplication and division)

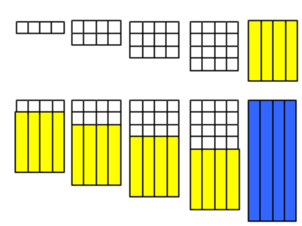
If you stack the skip counting chunks from the grids above, or the hops from a number line, you get a rectangular array These rectangular arrays are the basis of the area model of multiplication and division. Here are some arrays that show common math facts from the times tables:



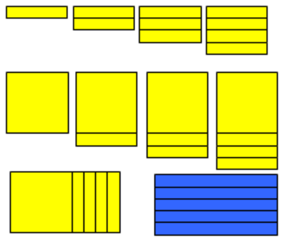
x 2



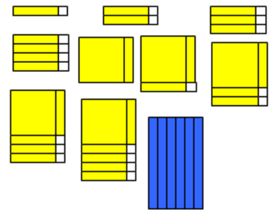
x 3



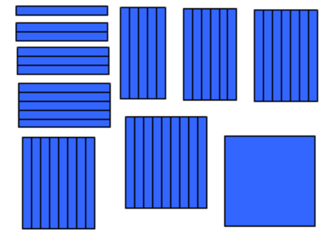
x 4



x 5

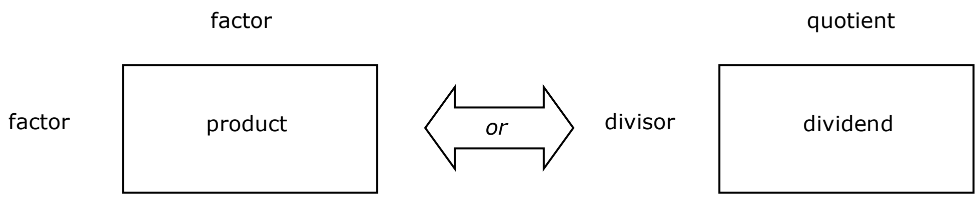


x 6



x 10

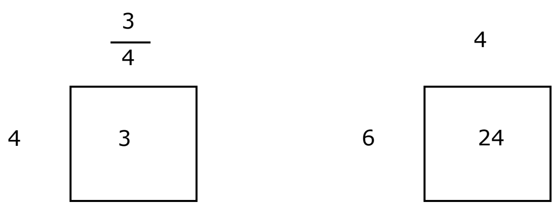
A rectangular array captures any multiplication or division relationship



Both of these are restatements of the same thing, since **m/n = k** means **m = n • k**

(Thanks to Professor Hung-Hsi Wu of UC Berkeley who led us through the derivation and exploration of these during a summer institute class of his we were lucky enough to be able to attend. See also his "Chapter 2: Fractions (Draft) at http://math.berkeley.edu/%7Ewu/EMI2a.pdf)

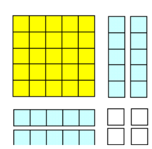
Here are illustrations of a couple of fractions 3/4 and 24/6:



multiplying and dividing

Multiplication and division should be taught together for the same reasons as with addition and subtraction, above.

arrays--4 facts: 8x2=16, 2x8=16, 16÷2=8, 16÷8=2
skip counting forwards & backwards (repeated addition and subtraction)
BUT YOU HAVE TO GET TO THE AREA MODEL BECAUSE REPEATED ADDITION
AND SUBTRACTION DON'T WORK WITH POLYNOMIALS
(unless you picture adding x repeatedly x times)
skip counting groups and bundles
stacking skip-counting steps into arrays (include 5x5 flats)
tables
fluency/automaticity: Holey Cards
gzinta ("goes-into") number x 1/n = 1 or n/n = 1 multiplicative inverse
n x 1 = n identity element for multiplication
exponents n^0, n^1, n^2, n^3, n^{-1}, n^{-2}, n^{-3}

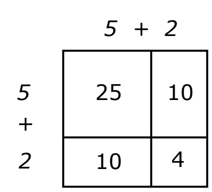


7 x 7 = 49

7 x 7 = 25 + 10 + 10 + 4 = 49

This area model for multiplication should be the goal for pre-secondary education.
(See Stanley Ocken's "Algorithms, Algebra, and Access" http://www.nychold.com/ocken-aaa01.pdf for strong reasons why the area model is critical for an understanding of multiplication and division.)

This leads right into the standard algorithm for multiple-digit multiplication:



(5 + 2) • (5 + 2) = 49

7 x 7 = 25 + 10 + 10 + 4 = 49

And the coming-together of the standard algorithm for multiplication, the area model for multiplication and division, and standard base 10 place value, leads right into multiplying polynomials—whether it be with base ten blocks, algebra tiles, foiling, or whatever:

$$\begin{array}{r} x + 2 \\ \times \quad x + 2 \\ \hline 2x + 4 \\ x^2 + 2x \quad\quad \\ \hline x^2 + 4x + 4 \end{array}$$

	x + 2	
x	x^2	2x
+		
2	2x	4

$(x + 2) \bullet (x + 2) = x^2 + 2x + 2x + 4$

$(x + 2)^2 = x^2 + 4x + 4$

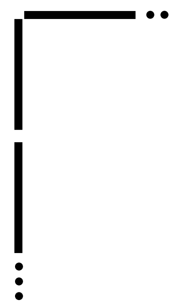
Consider 23 x 12:

$$\begin{array}{r} 12 \\ \times \; 23 \\ \hline 6 \\ 30 \\ 40 \\ 200 \\ \hline 276 \end{array}$$

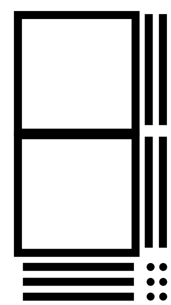
standard algorithm showing partial products

$$\begin{array}{r} 10 + 2 \\ \times \quad 20 + 3 \\ \hline 30 + 6 \\ 200 + 40 \quad\quad \\ \hline 200 + 70 + 6 \end{array}$$

seen as a polynomial



dimensioned with base 10 blocks



filled-in

	10 + 2	
20	200	40
+		
3	30	6

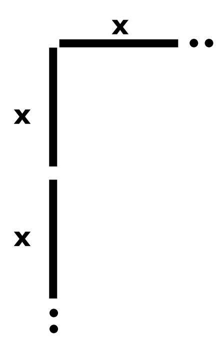
$(10 + 2) \bullet (20 + 3) = 276$

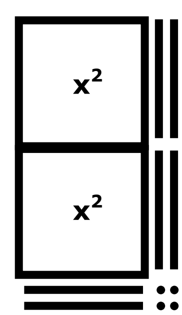
$12 \times 23 = 200 + 40 + 30 + 6 = 276$

seen in the not-to-scale polynomial grid

Now consider how similar is $(x + 2) \bullet (2x + 3)$:
This is why the area model is essential.)

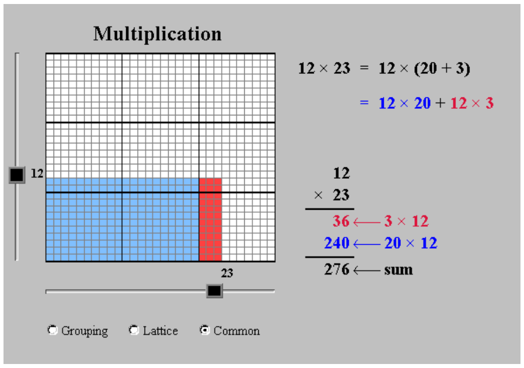
$$\begin{array}{r} x + 2 \\ 2x + 3 \\ \hline 3x + 6 \\ 2x^2 + 4x \quad\quad \\ \hline 2x^2 + 7x + 6 \end{array}$$



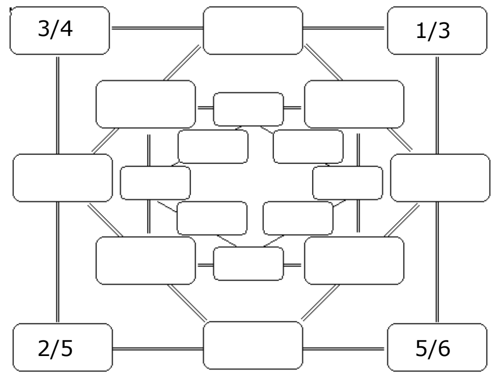


	x + 2	
2x	$2x^2$	4x
+		
3	3x	6

$(x + 2) \bullet (2x + 3) = 2x^2 + 4x + 3x + 6 = 2x^2 + 7x + 6$



For an online, interactive look at the area model, go to
the National Library of Virtual Manipulatives
http://nlvm.usu.edu/en/nav/frames_asid_192_g_2_t_1.html



diffy with fractions

fractions

- number line: rational numbers
- fractions of a whole
- fractions of a group
- number line: distance in steps taken/distance in steps to the goal
- arrays: percent or fraction of coverage
- common
- improper
- adding and subtracting: diffies
- common denominators
- reducing fractions
- mixed numbers
- decimals
- percents
- picturing
- equivalent fractions
- factoring: make a table and graph it; use Excel if you like (see the example in graphing x-y coordinates, below)
- chance (what are the chances of rolling a 1? a 6, a 9?)
  - what is the chance that I will win? that somebody will win?
  - what is the chance that I will roll a 39? (a 3 on one die, a 9 on the other)
  - what is the chance that Valentine’s Day will be on a Sunday?
- n/n = 1/1 = 1 = 100/100 = 100%

to give students an idea of what’s happening in multiplication and division of fractions, a table works very well:

3x = y		
	x	y
3	4	12.00
3	2	6.00
3	1	3.00
3	0.50	1.50
3	0.25	0.75

as x gets smaller, y gets smaller

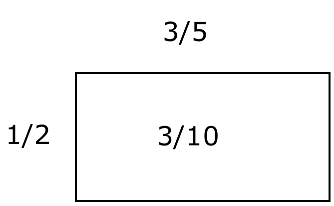
It’s pretty easy to see that when you multiply 3 by smaller and smaller numbers, you get smaller and smaller answers. Particularly, when you multiply it by less than 1, you get less than 3.

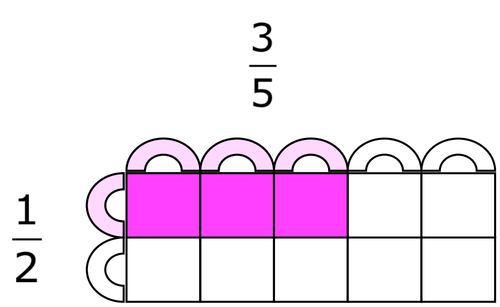
And it’s pretty easy to see that when you divide something by smaller and smaller numbers, you get bigger and bigger results.

3/x = y		
	x	y
3	4	0.75
3	2	1.50
3	1	3.00
3	0.50	6.00
3	0.25	12.00

as x gets smaller, y gets bigger

And, of course, since it’s multiplication and division,
3/5 x 1/2 can be pictured as a rectangular array:





dimensions (0, 1, 2, 3)

units (counting, distance, area, volume)
- appropriate units
- unit equations

distance around (perimeter) vs. area covered string (BL) and 3″ x 5″ index cards (BL)

arrays (covering)
- area of rectangles and squares
- area of parallelograms and rhombi
- area of trapezoids
- area of triangles
- area of a unit circle—use 3 x 5 card method (BL)

volume (filling)

mapping, scaling, and modeling (distance and area)
- the classroom, the school
- use Google maps to see this big-time and online

> Area always has something to do with base x height (length x width) and any triangle will be half of a parallelogram.
>
> So really, there's just b*h (and b*h/2 for triangles) and πr^2 for circles

reading and writing sentences (linear equations)

2 + 2 = 4
2 + x = 4
x + y = 4

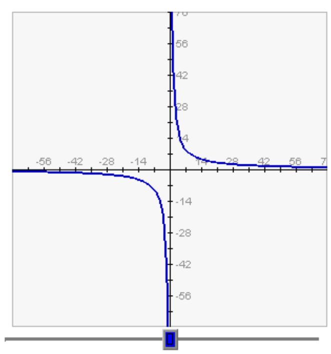
tables

x and y values
multiplying by fractions (x 4, x 3, x 2, x 1, x ½, x 1/3, x ¼)
possible outcomes (rolls of a dice)

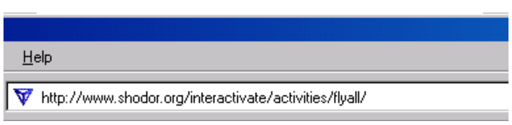
graphing x-y coordinates

Basic operations (addition, subtraction, multiplication, division) should be pictured on coordinate axes. This is one way into higher math. Probably on the first pass, teachers will want to omit the negative numbers and stay in the first quadrant. On the second pass, the negative numbers should certainly be included as well as fractional values for x and y. Excel does fine tables (below) and okay graphs, but for good graphs, go to http://www.shodor.org/interactivate/activities/flyall/

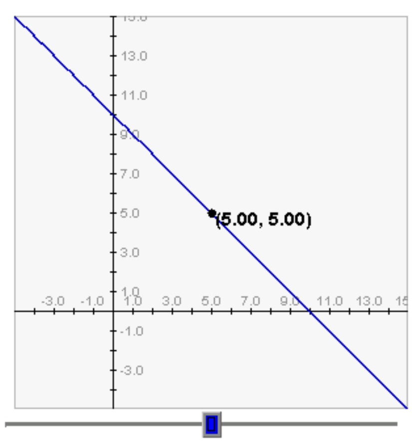
x	y
xy = 63	
63	1
21	3
7	9
3	21
1	63
-63	-1
-21	-3
-7	-9
-3	-21
-1	-63



f(x) = 63/x

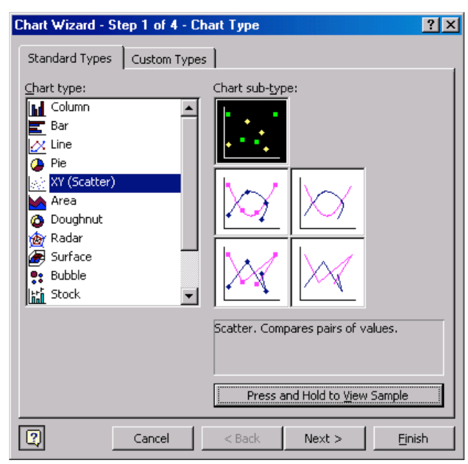


x	y
x + y = 10	
0	10
1	9
2	8
3	7
4	6
5	5
6	4
7	3
8	2
9	1
10	0



f(x) = 10-x

> It's easier on the eye, I think, if you tell Excel to center the values and if you merge the cells in the second row and put the equation in there.
>
> For the graph, I generally like to use Scatter.



constructions

Constructions with compass and straightedge will give everybody a visceral, kinesthetic connection with geometry. It probably would be a good idea for students (and teachers) to write down each step on the side.

- line segment (connecting two points)
- circle
- copy line segment
- perpendicular bisector of a line segment
- angle
- copy angle
- bisect angle
- triangle from three given lengths, as long as the two shortest ones put end-to-end are longer than the longest
- square
- line parallel to a given line
- square with circumscribed circle
- square with inscribed circle
- triangle with circumscribed circle
- triangle with inscribed circle

stylistic protocols: show your work (download the PowerPoint from www.soesd.k12.or.us/Page.asp?NavID=776)

Work that's not done along these lines simply doesn't meet class standards and is therefore incomplete and can't be accepted.

- draw a diagram,
- skip lines or leave a space between lines,
- box your answer,
- check your work

boardwork protocols

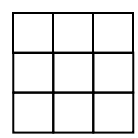
All of us need to know how to explain our stuff. Some people are shy; some love the limelight. Whatever. All students should have to do work at the board, but we need to set them up for success: they should do a problem they have worked on and polished with a peer coach, making sure that they follow the stylistic protocols above.

problem solving heuristics and mnemonics

- show what you know: draw a diagram
- list what you know: define variables
- make a table of values
- use movement to remember
    - open your hands to 30°, 120°, 90°
    - turn through 180°
    - put your elbows together and form a 90° angle with your forearms

conventions

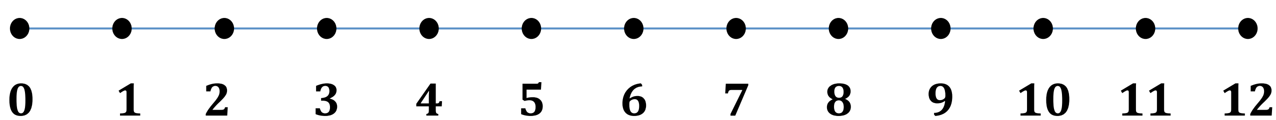
- $2\frac{3}{4}$ means $2 + \frac{3}{4}$ but 23 means $20 + 3$ and $2x$ means "two times x", i.e., $x + x$
- parentheses for grouping and multiplication
- * means multiplication, also x and • and ( )
- / means division, sometimes separates numerator and denominator
- order of operations
- superscript: 3^2 means three squared
- 3x3, 3•3, 3*3, and 3^2 all mean 3 times itself 2 times, i.e., 3^2, pronounced "three squared"
- x = "x" or one times x; 3x = "three x" or three times x.



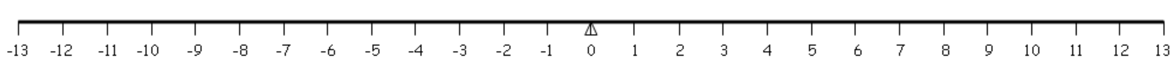
3 squared looks like this

number lines

number line: natural numbers (include zero)



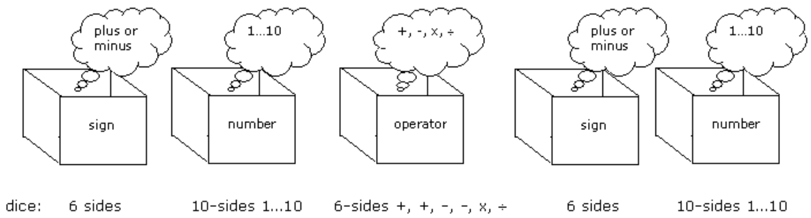
number line: integers (left and right)



number line: rational numbers (fractions)
number line: real numbers (roots, plus transcendental numbers like pi and e)
coordinate axes are just number lines rotated 90° around zero

dice

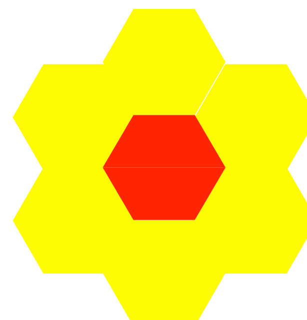
Rolling dice is a good way for students to create—and own!—their own problems. Using dice also keeps problems from being too predictable and boring and keeps problems robust enough to allow—and force!—students to generalize. A bad side effect of many worksheets is that students get trained in an algorithm rather than gaining a deeper understanding of the problem space. Dice are an important part of race games and diffies. Here's an example that uses three kinds of dice, five in all, that creates a sum, remainder, product, or quotient in the rational numbers:



- dice with + or minus on each of six faces
- dice with add, subtract, multiply, or divide on each of six faces
- dice with 1…10 or 0…9 on each of ten faces

- dice with 1/2, 1/3, 2/3, 1/6, 5/6, and 0 on each of 6 faces
- dice with 1/2, 1/3, 2/3, 1/6, 5/6, 1/12, 5/12, 7/12, 11/12, 1/4, 3/4, and 0 on a 12-sided die—or just use the 6-sided die above with halves, thirds, sixths, and zero along with another 6-sided die with the remaining fourths and twelfths.

You could use these to race for a pattern block flower or even for a couple of egg cartons.
For more about race games, see "adding and subtracting", above and an overview of race games at
http://www.soesd.k12.or.us/files/race_games.pdf
And you can download virtual 10-sided dice from http://www.soesd.k12.or.us/Page.asp?NavID=741



data management (statistics, probability, chance)

Students—and classrooms!—should keep track of how they're doing, set targets, evaluate their progress, and troubleshoot difficulties. They can use dice to build their own problems (see above) and begin to get an appreciation for how chance works.

resources

SOESD's math pages at www.soesd.k12.or.us/math and math resources at www.soesd.k12.or.us/math/math_resources

These are the best places I've found on the web for javascript problemspaces:

National Library of Virtual Manipulatives: http://matti.usu.edu/nlvm/nav/vlibrary.html
Project Interactivate: http://www.shodor.org/interactivate/
Links to my favorites, along with short explanatory blurbs are at http://www.soesd.k12.or.us/files/manipulinks.pdf
Templates, handouts, rants, and overviews are at SOESD's Math Resources page www.soesd.k12.or.us/math
PowerPoints and MS Office graphics techniques are at SOESD's training page, www.soesd.k12.or.us/support/training

There are some essays online, that will inform and provoke your thinking. Here are some of my favorites:

Howe, Roger. "Taking Place Value Seriously: Arithmetic, Estimation and Algebra". http://www.maa.org/sites/default/files/pdf/pmet/resources/PVHoweEpp-Nov2008.pdf

Ocken, Stanley. "Algorithms, Algebra, and Access". http://www.nychold.com/ocken-aaa01.pdf

Wu, Hung-Hsi. "How to Prepare Students for Algebra", *American Educator*, Summer 2001, Vol. 25, No. 2, pp. 10-17. http://www.aft.org/pubs-reports/american_educator/summer2001/index.html

Wu, Hung-Hsi. "Introduction to School Algebra [DRAFT]". http://math.berkeley.edu/~wu/Algebrasummary.pdf (Wu's homepage http://math.berkeley.edu/~wu/ has many other interesting essays that can be downloaded in pdf format.)

Wu, Hung-Hsi. "Key Mathematical Ideas in Grades 5-8". http://math.berkeley.edu/~wu/NCTM2005a.pdf

Wu, Hung-Hsi. "Chapter 2: Fractions (Draft)". http://math.berkeley.edu/%7Ewu/EMI2a.pdf

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